Tree-Cylinders
(what they can do for you!)

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12 May 2018
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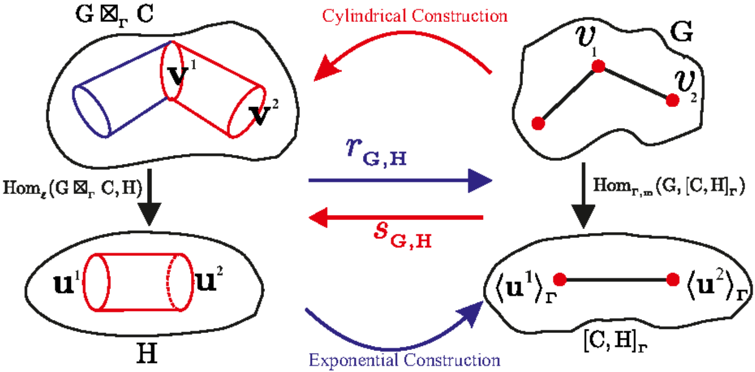
Outline

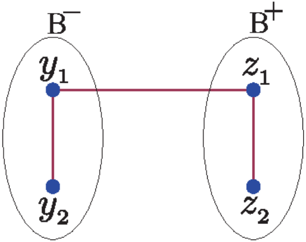
1. Cylindrical constructs
   - Some examples
   - π-lifts vs. 2-lifts
2. Sparsifying the clique
   - Regular random π-lifts
   - Tree-cylinders
   - A class of highly symmetric graphs
   - On Reed's conjecture for sparse graphs
3. Concluding remarks

Schematic Duality Diagram

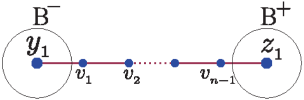


Note: These do not necessarily give rise to topological lifts!

A couple of cylinders

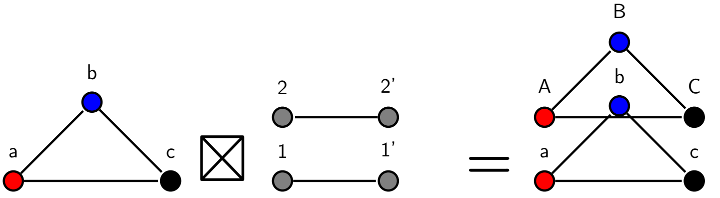


The π-cylinder,



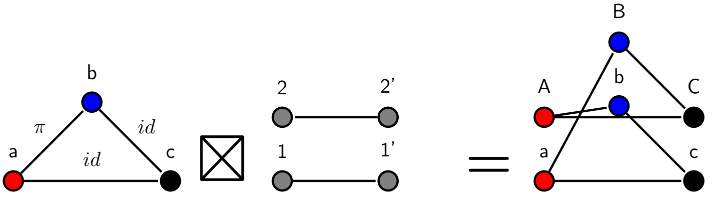
Path cylinder P_n

2-lifts



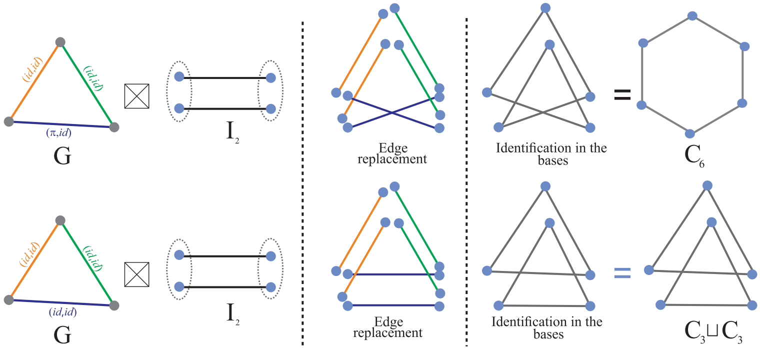
$$\begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array} \begin{matrix} a \\ b \\ c \end{matrix} \boxtimes \begin{array}{c} \begin{matrix} 1 & 2 \end{matrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \begin{matrix} 1 \\ 2 \end{matrix} = \begin{array}{c} \begin{matrix} a & A & b & B & c & C \end{matrix} \\ \left(\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{array}\right) \end{array} \begin{matrix} a \\ A \\ b \\ B \\ c \\ C \end{matrix}$$

2-lifts



$$\begin{matrix} & a & b & c \\ a & 0 & \pi & 1 \\ b & \pi & 0 & 1 \\ c & 1 & 1 & 0 \end{matrix} \boxtimes \begin{matrix} & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{matrix} = \begin{matrix} & a & A & b & B & c & C \\ a & 0 & 0 & \mathbf{0} & \mathbf{1} & 1 & 0 \\ A & 0 & 0 & \mathbf{1} & \mathbf{0} & 0 & 1 \\ b & \mathbf{0} & \mathbf{1} & 0 & 0 & 1 & 0 \\ B & \mathbf{1} & \mathbf{0} & 0 & 0 & 0 & 1 \\ c & 1 & 0 & 1 & 0 & 0 & 0 \\ C & 0 & 1 & 0 & 1 & 0 & 0 \end{matrix}$$

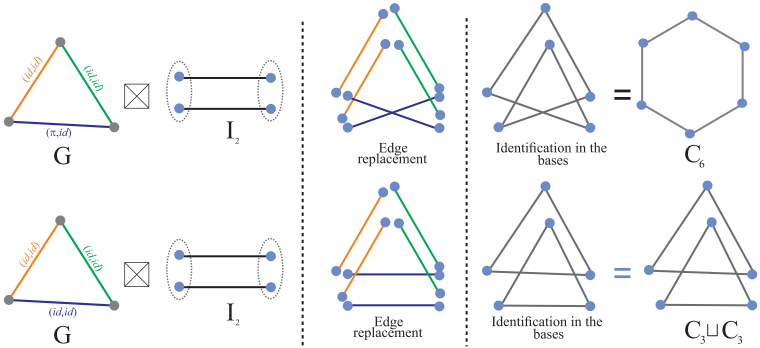
2-lifts


(id,id)
(id,id)
(π,id)
G
I2
Edge replacement
Identification in the bases
=
C6
(id,id)
(id,id)
(id,id)
G
I2
Edge replacement
Identification in the bases
=
C3⊔C3

Some examples

A directed cylindrical construction	Show(C_{12})
Cartesian product as a cylindrical construction	Show($C_3 \Box P_2$)
The Petersen graph	Show
Subdivision and powers	Show
The exponential graph $[K_2 \Box K_2, K_3]$	Show

2-lifts


(id,id)
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=
C3⊔C3

Random 2-lifts

Main question

To what extent random 2-lifts preserve connectedness?

Fact

The spectrum of a 2-lift of a graph G contains the spectrum of G as well as some new eigenvalues.

Conjecture [Y. Bilu and N. Linial 2004+]

Every d-regular graph has a 2-lift whose new eigenvalues have absolute value less than or equal to $2\sqrt{d-1}$.

Note: If this is true we get Ramanujan graphs starting from complete graphs!

Random 2-lifts

A. W. Marcus, D. A. Spielman, N. Srivastava, 2013+
Published in *Annals of Mathematics* (2015)

There exist (arbitrarily large enough) bipartite regular Ramanujan graphs of arbitrary degree.

The proof is based on the fundamental technique of interlacing families of polynomials which is also used by the same authors to prove Kadison-Singer Problem.

Lifts and Randomness: MSS sketch of proof

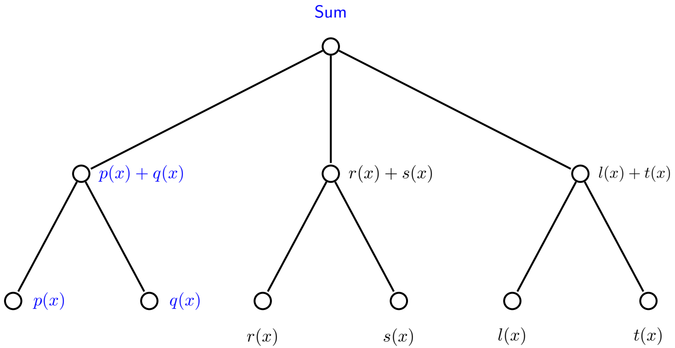
Let $\tilde{G}$ be a typical random 2-lift of a d-regular Ramanujan graph G.

- $\chi(\tilde{G}) = \chi(G)\lambda(\tilde{G})$.
- Prove that $\{\lambda(\tilde{G})\}$ is an interlacing family of polynomials.
- The mean of $\{\lambda(\tilde{G})\}$ is the matching polynomial.
- Then there exists a lift for which

$$Maxroot\ \lambda(\tilde{G}) \leq Maxroot\ matchingpoly \leq 2\sqrt{d-1}.$$

- Use bipartiteness to impose symmetry on the spectrum!

Interlacing families of polynomials



If siblings are interlacing then there exists a leaf as L such that

$$Maxroot(L) \leq Maxroot(Sum).$$

Main question

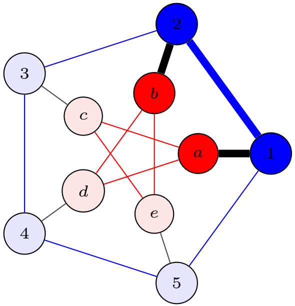
- It seems that 2-lifts preserve connectedness.
- But 2-lifts also preserve the degree.

Question

Is it possible to use random lifts to construct sparsifiers that reduce the degree but also preserve connectedness?

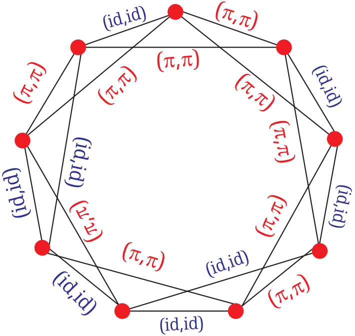
If the answer is YES then one may start from the complete graph K_n as the most connected graph and sparsify to sparser graphs of smaller degrees!

π-lifts

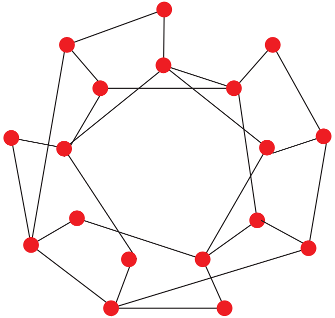


$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

A random π-lifts

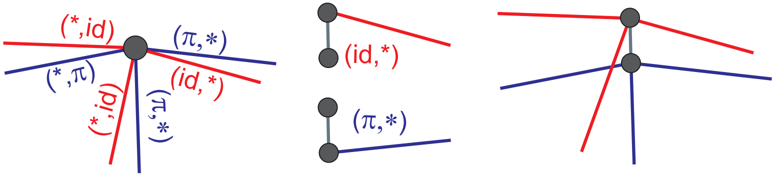


(a)



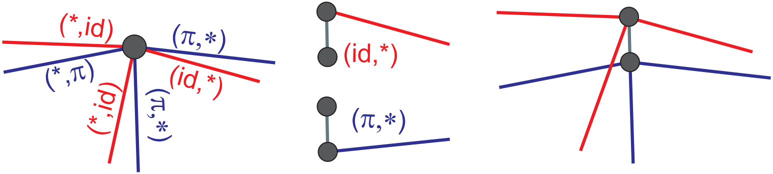
(b)

Regular random π-lifts: construction



- Consider a $(d = 2k)$-regular graph
- For each vertex v, choose half of the edges at random and assign the label π to them. Assign the label id to the rest of the edges attached to v.
- Construct the corresponding π-lift from the labeled-graph which is a $(k + 1)$-regular graph.

Iterated Random π-lifts: an important ensemble



- Start: The complete graph K_{2^t+3}.
- Sample: Iteratively, apply random π-lifts for t stages.
- Outcome: Is a 3-regular element of the ensemble.

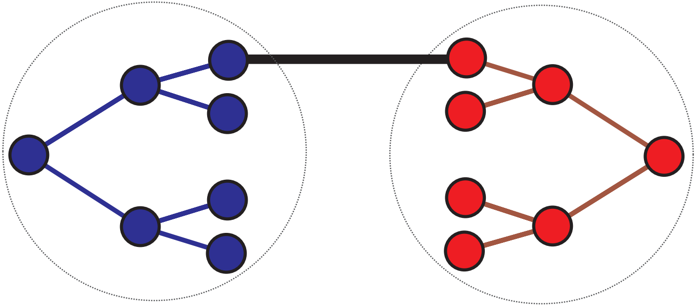
Main idea

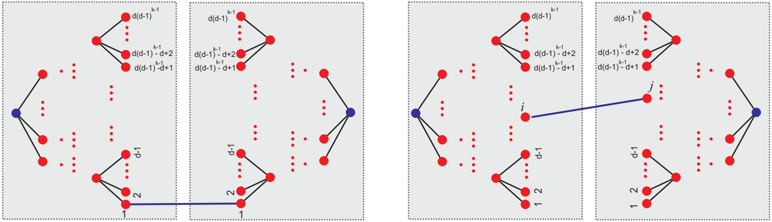
Roadmap:

Find a cylinder that
sparsifies, preserves connectivity and reduces the degree at the same time.

Then sparsify the complete graph using this cylindrical construction!

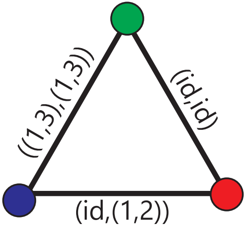
Tree-cylinders: how they help?

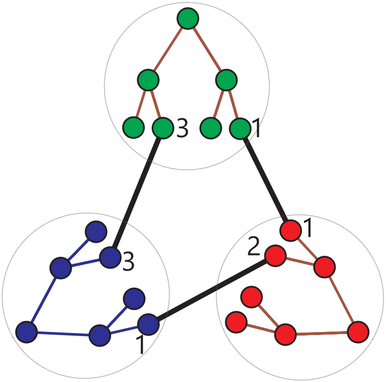




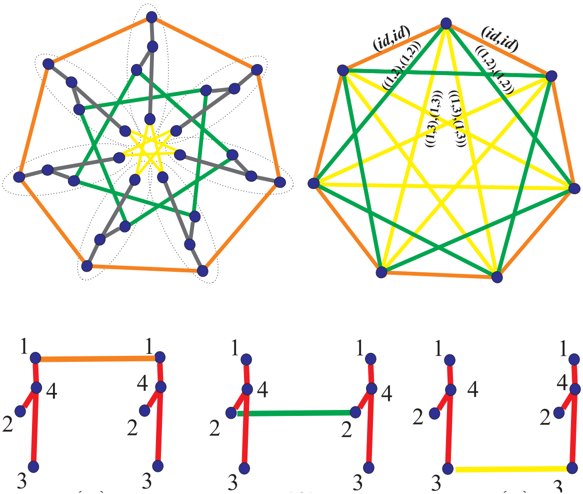
(a) (b)

Tree-cylinders: how they help?

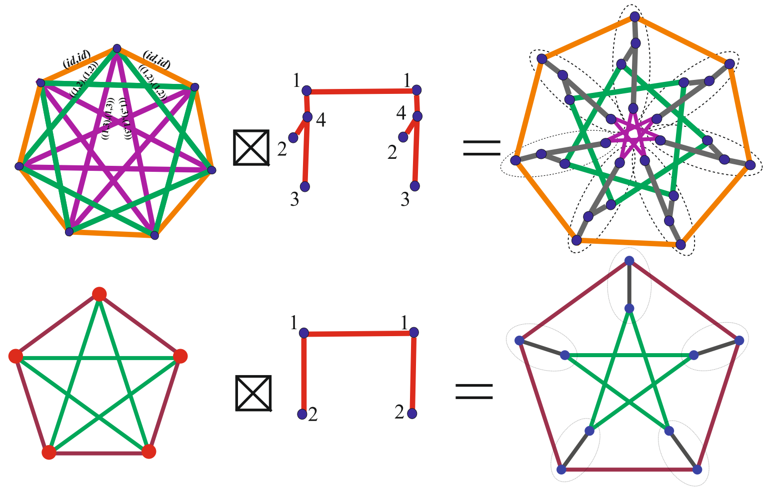




Tree-cylinders (the Coxeter graph)



Some tree-lifts of complete graphs



Random tree-lifts: a challenge for small degrees

[I. Dumitriu and S. Pal 2012]

The empirical distribution of the eigenvalues of adjacency matrices of sparse regular random graphs converges to the semicircle law, when the degree slowly increases to infinity with the number of vertices.

Note: The ensemble is the whole class of labeled d-regular graphs with uniform distribution.

Challenge!

Can we sample in a better and more efficient way?
Candidate: The ensemble of random tree-lifts!

Random tree-lifts: main questions

- Study the basic combinatorial features of the ensemble (e.g. girth, independence number, Hamiltonicity, chromatic number,...)
- Study connectedness in the mean (is there an improvement?)
- Study the spectral gap as well as the hard edge.
- Study these properties in the limit and note that the order grows exponentially.
- Study this ensemble as an ensemble of random $\{0, 1\}$-matrices.
- Try to construct extremal species.

T-graphs

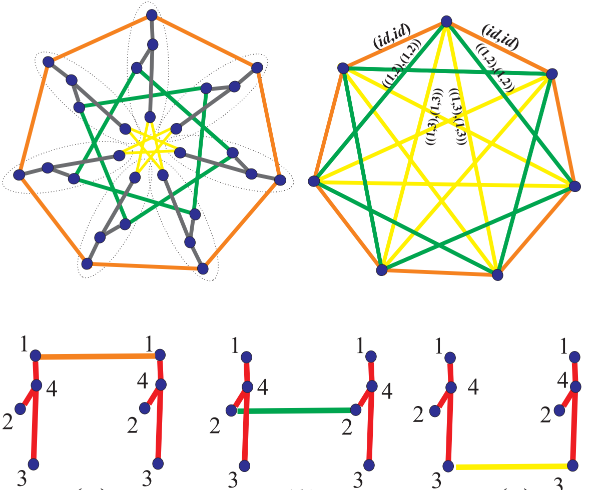
History

Tree cylinders of M. Madani + Group labeling of A. Taherkhani $\Rightarrow$ T-graphs!

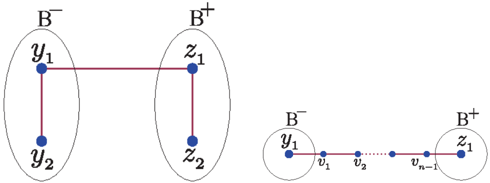
Definition

A T-graph is a tree-lift that can be described as replacing each vertex of a complete graph by a complete tree and join the leaves in a special predefined order called group labeling of trees.

Examples of T-graphs (the Coxeter graph)



Bilateral Symmetry and Commutative Decompositions



The π-cylinder, Path cylinder P_n

Bsymmetric Cylinders (informal!)

A cylinder is bsymmetric if it has a symmetry with respect to its ends (i.e. bases).

Commutative decompositions

For $t \geq 1$, a *commutative t-decomposition* of a graph G is a family of t edge-disjoint spanning subgraphs of G, as $(\mathrm{G}_0, \cdots, \mathrm{G}_{t-1})$ each of which has no isolated vertex such that for every pair of disjoint indices i and j, the matrices G_i and G_j commute, and moreover, $G = \sum_{i=0}^{t-1} G_i$.

Bilateral Symmetry, Commutative Decompositions and Spectrum

Let $\mathcal{H}$ be a symmetric cylinder with no internal vertices (e.g. a tree-cylinder), then

A spectral result

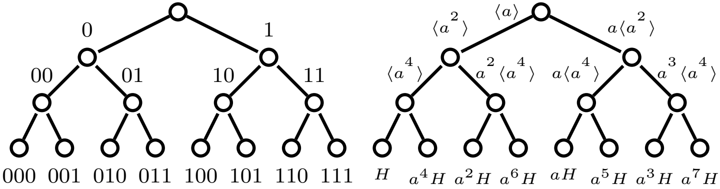
$$\chi(\mathcal{G} \boxtimes \mathcal{H}, x) = \prod_{j=1}^{n} \chi\left(T + \sum_{i=0}^{t-1} \theta_i^j E_i^{bb'}, x\right),$$

in which, T is the base of the cylinder, χ is the characteristic polynomial and sum is a term depending on the commutative decomposition.

Summary!

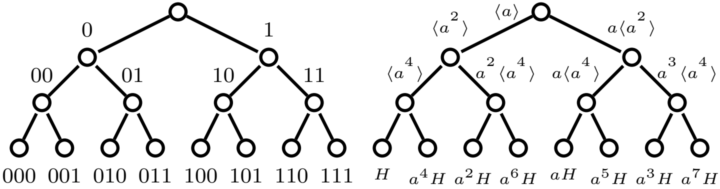
The spectrum of such a construction is a perturbation of the spectrum of the base depending on the construction and the twists.

Magic labeling



- An example of cyclic group labeling of a binary tree T_3 by $\Gamma = \langle a \mid a^{16} = 1 \rangle$.
- $H \stackrel{\text{def}}{=} \langle a^8 \rangle$.
- if $s = s_m \cdots s_0$ is a binary string of length $m+1$, then its *reverse* is defined as $\hat{s} \stackrel{\text{def}}{=} s_0 \cdots s_m$ and its *corresponding integer* is defined as $(s)_2 \stackrel{\text{def}}{=} s_m 2^m + \cdots + s_1 2 + s_0$.

Magic labeling



- Let $\Gamma = \langle a \mid a^N = 1 \rangle$ be a multiplicative cyclic group of order N generated by a, and let $\mathcal{C}(\Gamma)$ be the whole class of cosets of subgroups of Γ. Then, the mapping $\gamma : V(\mathrm{T}_h) \to \mathcal{C}(\Gamma)$ defined as

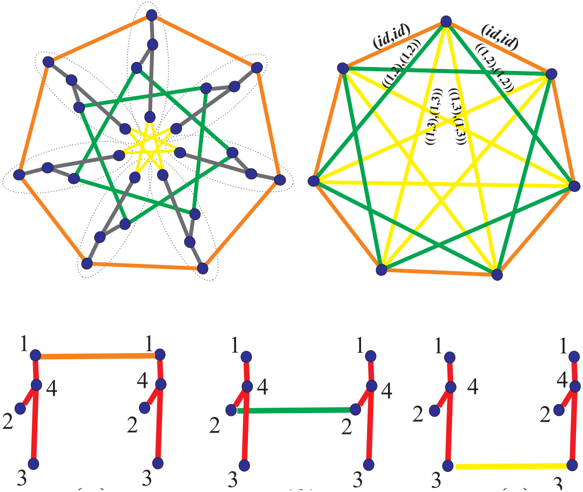
$$\gamma(s) \stackrel{\text{def}}{=} a^{(\hat{s})_2} \langle a^{2^{|s|}} \rangle$$

is called a cyclic group labeling of T_h by Γ.

A class of highly symmetric graphs

- Let $t = 3 \times 2^{h-1}$ such that $n = 2t + 1$ is a prime number. Consider the multiplicative group $\Gamma = \mathbb{Z}_n^*$ of the field $\mathbb{Z}_n$ and let a and b be its generators for which $b^3 = 1$ and $a^{2t/3} = 1$.
- Label the root of $\mathrm{T}^{\bullet}_{3,h}$ by Γ itself and label each of its siblings as s with $|s| = 1$ by the cosets $b^s\langle a \rangle$. Also, let γ be the cyclic group labeling of a complete binary tree of height $h - 1$ under the vertex 0 by the cyclic group $\langle a \rangle$ and for the subtrees below the vertices 1 and 2 apply translation (i.e. multiplying the cosets) by b and b^2, respectively, of the labeling γ of the subtree under 0.
- Again, for the leaf i, let $\gamma(i)^*$ be the element of the coset $\gamma(i) = \{\gamma(i)^*, -\gamma(i)^*\}$ which is greater than or equal to 1 and less than or equal to t, and define $k_i \stackrel{\text{def}}{=} \gamma(i)^*$.

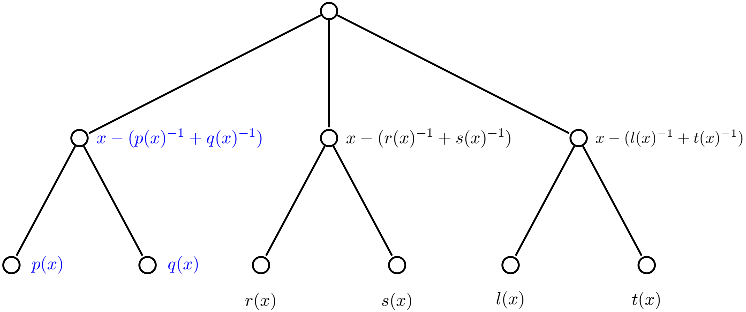
Tree-cylinders (the Coxeter graph)



Here: $h = 1, t = 3, b^3 = 1, a^2 = 1.$

Eigenvalue Mixing

do the same!



This is essentially how the determinant of a perturbation of a tree can be computed in most important cases!

A 3-regular Ramanujan graph of order 130

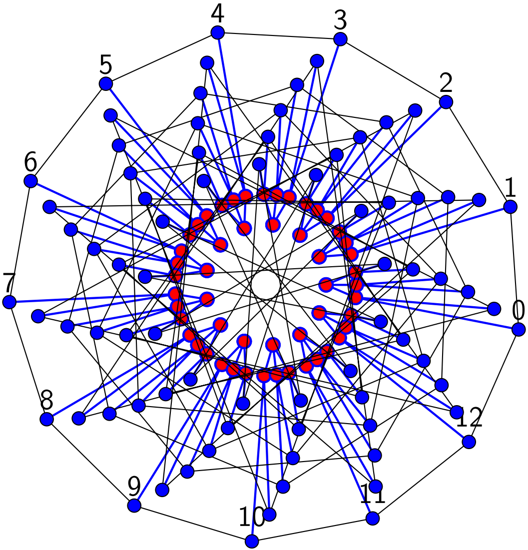
Setup

Take the 3-regular tree of hight 2 with 6 leaves as the base of the tree-cylinders and choose the complete graph on 13 vertices as the base-graph of the construction.

Using group-labeling this gives rise to a 3-regular Ramanujan graph of order 130 with the following characteristic polynomial,

$$\phi(\mathcal{K}_{13} \boxtimes \mathcal{H}^{\bullet}, x) = (x-3)(x-1)(x+2)(x-2)^3(x^2-2x-2)^2$$
$$\times(x^{10}+x^9-14x^8-12x^7+65x^6+45x^5-115x^4-55x^3+69x^2+12x-10)^{12}.$$

Roots:
$[-2.635(12), -2.197(12), -2.000, -1.603(12), -1.135(12), -0.732(2), -0.485(12), 0.396(12), 0.670(12), 1, 1.424(12), 2(3), 2.08(12), 2.485(12), 2.732(2), 3.000]$


0
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12

Chromatic number under sparsity I

A conjecture [B. Reed 1998]

$\chi(G) \leq \lceil \frac{1}{2}(\Delta + \omega + 1) \rceil.$

Main question

How does χ behave under structural sparsity conditions?

Reed's conjecture for triangle-free graphs

$\chi(G) \leq \frac{\Delta}{2} + 2.$

Chromatic number under sparsity II

Some facts

- Conjecture is true for $\Delta \leq 4$ (Consequence of Brooks' theorem).
- [A. V. Kostochka (1978)]: If G has large girth and $\Delta(G) \geq 5$ then $\chi(G) \leq \frac{\Delta}{2} + 2$.
- A. Johansson (1996): For triangle-free graphs with large Δ we have $\chi(G) \leq O(\frac{\Delta}{\log \Delta})$.

Kostochka and Reed parameters

Let $\mathcal{G}'(d, g)$ be the class of all graphs of maximum degree d and of girth larger than g. Define,

$$R(d) \stackrel{\text{def}}{=} \sup_{g>3} \sup_{G \in \mathcal{G}'(d,g)} \chi(G),$$

$$K(d) \stackrel{\text{def}}{=} \inf_{g>3} \sup_{G \in \mathcal{G}'(d,g)} \chi(G).$$

Chromatic number under sparsity III

Kostochka-Reed Conjecture

For any triangle-free graph we have $\chi(G) \leq \frac{\Delta}{2} + 2$; i.e., $\forall\ d \geq 2 \quad R(d) \leq \frac{\Delta}{2} + 2.$

The first open questions!

Although the conjecture is correct asymptotically and also when $\Delta \leq 4$, it is still open for small values of Δ. The first open cases are:

A: Does there exists a 5-chromatic triangle-free graph with $\Delta \leq 5$?

B: Does there exists a 6-chromatic triangle-free graph with $\Delta \leq 6$?

- B seems to be the easiest open case!

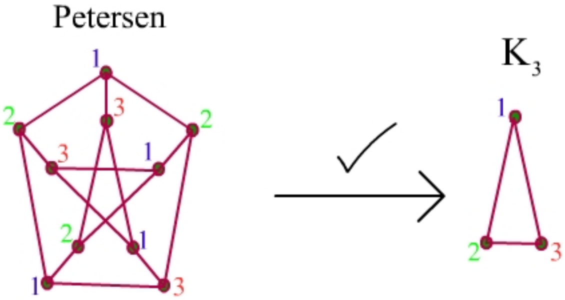
Chromatic number under sparsity III

Idea

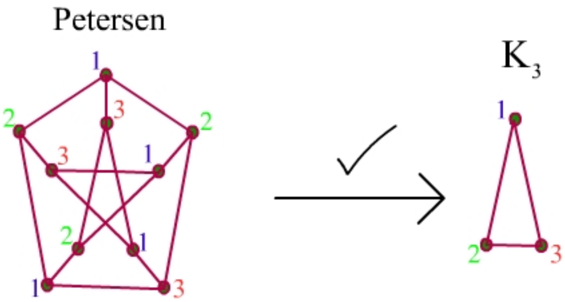
What if we impose more constraints on the coloring problem itself?

Let's talk about this!

Graph colouring

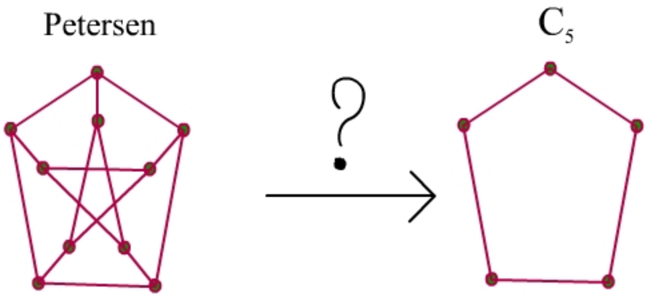


Graph colouring

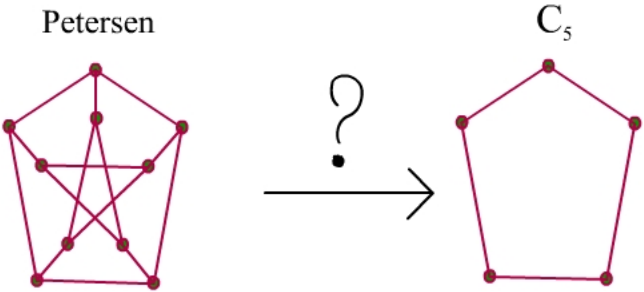


- Homomorphisms to K_n is equivalent to colouring the vertices of the graph by n colours such that the terminal ends of each edge have different colours.

Another question!



Another question!



- Does there exist a homomorphism from the Petersen graph to the 5-cycle C_5?

Circular chromatic number

- The circular complete graph, $K_{\frac{n}{r}}$, has the vertex set

$$\{0, 1, \cdots, n-1\}$$

and the edge set

$$\{ij \mid r \leq |i-j| \leq n-r\}.$$

- The circular chromatic number of a graph G is defined as

$$\chi_c(G) \stackrel{\text{def}}{=} \inf\{\frac{n}{r} \mid G \longrightarrow K_{\frac{n}{r}}\}.$$

- Example: $K_{\frac{5}{2}} = C_5$.

Girth-closed classes

Girth-closed and odd girth-closed classes

A class of simple graphs $\mathcal{G}$ is said to be girth-closed (resp. odd-girth-closed) if for any positive integer g there exists a graph $G \in \mathcal{G}$ such that the girth (resp. odd-girth) of G is greater than or equal to g.

Pentagonal and odd-pentagonal classes

A girth-closed (resp. odd-girth-closed) class of graphs $\mathcal{G}$ is said to be pentagonal (resp. odd-pentagonal) if there exists a positive integer g^* depending on $\mathcal{G}$ such that any graph $G \in \mathcal{G}$ whose girth (resp. odd-girth) is greater than g^* admits a homomorphism to the five cycle (i.e. is C_5-colorable).

The pentagon problem

Nešetřil parameter

Let $\mathcal{G}(d, g)$ be the class of all d-regular graphs of girth larger than g. Define,

$$Nes(d) \stackrel{\text{def}}{=} \inf_{g>3} \sup_{G \in \mathcal{G}(d,g)} \chi_c(G).$$

Pentagon problem [J. Nešetřil (1999)]

Is the class of simple 3-regular graphs pentagonal? i.e.,
Is it true that $Nes(3) \leq 2.5$?

Some negative results

- [A. V. Kostochka, J. Nešetřil, P. Smolikova (2001)]
  If C_5 is replaced by C_{11}, then a similar conjecture does not hold.
- [I. M. Wanless and N. C. Wormald (2001)]
  If C_5 is replaced by C_9, then a similar conjecture does not hold.
- [H. Hatami (2005)]
  If C_5 is replaced by C_7 then a similar conjecture does not hold.

Some positive results

- [Brooks' theorem] If C_5 is replaced by C_3, then a similar conjecture does hold.
- [A. Galluccioa *et.al.* (2001)] For every fixed simple graph H the class of H-minor free graphs is pentagonal.
- [O. V. Borodin *et.al.* (2004)] The class of planar graphs, projective planar graphs, graphs that can be embedded on the torus or Klein bottle are pentagonal.
- [O. V. Borodin *et.al.* (2008)] The class of simple graphs as G for which every subgraph of G has average degree less than $12/5$, is pentagonal (actually with $g^* = 3$).

The odd-girth constraint

Negative results

- [M. Gebleh (2007)]
  There exists an odd-girth-closed subclass of simple 3-regular graphs (i.e. spiderweb graphs) which is not odd-pentagonal. (Actually, the circular chromatic number of any spiderweb graph is equal to 3.)

Positive results

- [D., M. Madani (2015)]
  Let $\mathcal{C}$ be the subclass of the class of generalized Petersen graphs for which one of the following conditions hold.
  (a) $\mathrm{Pet}(n, k)$, where k is even, n is odd and $n \overset{k-1}{\equiv} \pm 2$.
  (b) $\mathrm{Pet}(n, k)$, where both n and k are odd and $n \geq 5k$.

  Then $\mathcal{C}$ is odd-pentagonal.

Chromatic number under sparsity III

Let's try to show that,

Objective

A lower bound for $K(d)$ implies a lower bound for $Nes(d)$.

Powers and subdivitions

Definitions

- The graph $G^{\frac{1}{t}}$ (i.e. the t-subdivision of G) is obtained by replacing each edge of G by a path of length t.
- The kth power functor on graphs is the right adjoint to the kth subdivision functor and for a graph G yields a graph G^k on the same vertex set where $u \sim v$ if there exists a walk of length k between u and v in G.
- For any graph G, define $G^{\frac{k}{t}} \stackrel{\text{def}}{=} (G^{\frac{1}{t}})^k$.

Note: If $G \longrightarrow H$ then for any integer $k > 1$ we have $G^k \longrightarrow H^k$.

Examples and applications

Examples

- $C_5^3 = K_5$.
- $Pet^3 = K_{10}$.
- For any integer $t \geq 1$ we have $(K_n)^{\frac{6t+1}{2t+1}} = K_{tn^2-tn+n}$.

Applications

- $\text{Petersen} \not\to C_5$.
- $\text{Coxeter} \not\to C_7$.

An implication of Pentagon problem (if the answer is YES)

For any integer g there exists a 3-regular graph of girth larger than g whose third power is 5-colorable.

Using tree-cylinders one can prove:

[M. Madani (2015)]

Let

- H be a graph with odd girth at least $2k+3$,
- G be a $d(d-1)^k$-regular graph with girth g and odd girth og, where $G \not\to H^{2k+1}$,

then there exist (many non-isomorphic) d-regular graphs G' such that

- $girth(G') \geq g$,
- $oddgirth(G') \geq og$,
- and $G' \not\to H$.

A connection to Reed's conjecture

A useful corollary

Let

- $(2k+1)r - nk > 0$,
- G be a $d(d-1)^k$-regular graph with girth g and odd girth og, where $\chi_c(G) > \frac{n}{(2k+1)r-nk}$,

then there exist (many non-isomorphic) d-regular graphs G' s.t.

- $girth(G') \geq g$,
- $oddgirth(G') \geq og$,
- and $\chi_c(G') > \frac{n}{r}$.

Sketch of the proof

- One should try to find a suitable labeling such that $G' \simeq G \boxtimes_{S_t} T_{d,k}$.
- First show that there exists an ordinary homomorphism

$$[T_{d,k}, H]_{S_t} \longrightarrow [P_{2k+1}, H] \simeq H^{2k+1}.$$

- Note that

$$G' \simeq G \boxtimes_{S_t} T_{d,k} \longrightarrow H$$
$$\Rightarrow \quad G \longrightarrow [T_{d,k}, H]_{S_t} \longrightarrow H^{2k+1}.$$

- Apply a result of Hajiabolhassan and Taherkhani indicating that for any non-bipartite graph G if $2K + 1 < og(K_{n/r})$ we have

$$K_{n/r}^{2k+1} \simeq K_{\frac{n}{(2k+1)r-kn}}.$$

A consequence

Summary!

$$K(d(d-1)^k) > \lceil \frac{n}{(2k+1)r - nk} \rceil \;\Rightarrow\; Nes(d) > \frac{n}{r}.$$

Let $d = 3$, $n = 5$, $r = 2$ and $k = 1$. Then, using this result,

A failure!

Existence of 6-regular 6-chromatic graphs of large girth would have disproved the pentagon problem!

But this is impossible since $K(6) \leq 5$!!!!

A second attempt

Let $d = 3$, $n = 7$, $r = 3$ and $k = 2$. Then, using this result,

A problem

$8 \leq K(12)$ implies $\frac{7}{3} < Nes(3)$.

This provides a constructive proof of H. Hatami's result!

Epilogue: some open problems

- Study the random π-lift model and prove that there exists π-lifts which are at least as connected as the base graph.
- Try to define a good and computable model for random tree-lifts.
- Study the extremal properties of T-graphs and their spectra.
- Construct a class of arbitrarily sparse 24-regular graphs of chromatic number larger than 9.
- Construct a class of arbitrarily sparse 12-regular graphs of chromatic number 8.
- Construct a triangle-free 6-regular graph of chromatic number 6.
- Construct a triangle-free 5-regular graph of chromatic number 5.

Epilogue: A quotation! (translation by Alain Connes)

In the prelude of "Récoltes et Semailles", Alexandre Grothendieck makes the following points on the search for relevant geometric models for physics and on Riemann's lecture on the foundations of geometry.
It must be already fifteen or twenty years ago that, leafing through the modest volume constituting the complete works of Riemann, I was struck by a remark of his "in passing".

... it could well be that the ultimate structure of space is discrete, while the continuous representations that we make of it constitute perhaps a simplification (perhaps excessive, in the long run ...) of a more complex reality; That for the human mind, "the continuous" was easier to grasp than the "discontinuous", and that it serves us, therefore, as an "approximation" to apprehend the discontinuous.

Thank you!

Comments are Welcome

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